

1.2.1.

Problemas con Cond. inicial

$$u' = f(t, u) \quad // \quad u(t_0) = u_0$$

$$\textcircled{*} u' = a u \rightarrow \left[u(t) = A_0 e^{at} \right]$$

$$\textcircled{*} u' = f(u) g(t) \rightarrow \left[\frac{du}{f(u)} = \int g(t) dt \right]$$

Sep.

$$\rightarrow u(t) =$$

$$- u' = f(au + bt) \quad z = au + bt$$

$$\frac{dz}{dt} = a \frac{du}{dt} + b \rightarrow \frac{du}{dt} = \left(\frac{dz}{dt} - b \right) \frac{1}{a}$$

$$\rightarrow \left(\frac{dz}{dt} - b \right) \frac{1}{a} = f(z)$$

$$\rightarrow \frac{dz}{dt} = af(z) + b \rightarrow \left[\int_{z_0}^z \frac{dz}{af(z) + b} = \int_{t_0}^t dt \right]$$

$$\textcircled{*} \frac{du}{dt} = f\left(\frac{u}{t}\right)$$

$$z = \frac{u}{t} \rightarrow \frac{du}{dt} = \frac{d}{dt}(tz) = z + tz'$$

$$z + tz' = f(z)$$

$$\rightarrow z' = \left(f(z) - z \right) \frac{1}{t}$$

$$\rightarrow \text{Sep.} \quad \int \frac{dz}{f(z) - z} = \int \frac{dt}{t}$$

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⊛ dif. totales

$$\frac{du}{dt} = \frac{-f(t, u)}{g(t, u)} \quad g \neq 0$$

$$\Leftrightarrow du \cdot g + f dt = 0 \longrightarrow d\phi(u, t) = \underbrace{\frac{\partial \phi}{\partial u}}_g du + \underbrace{\frac{\partial \phi}{\partial t}}_f dt$$

pero $\frac{\partial \phi}{\partial u \partial t} = \frac{\partial \phi}{\partial t \partial u}$

$$\rightarrow \text{si } \frac{\partial g}{\partial t} = \frac{\partial f}{\partial u} \rightarrow \exists \phi$$

$$\rightarrow \phi = \int g du + \psi(t)$$

$$\rightarrow C' \neq \frac{\partial}{\partial t} \int g du = f$$

$$\rightarrow C = \int f(t) dt - \int g du$$

⊛ factor int.

si $\frac{\partial g}{\partial t} \neq \frac{\partial f}{\partial u}$, no $\frac{\partial g}{\partial t} = \frac{\partial f}{\partial u}$

$$\rightarrow \text{si multiplico todo por } e^{\int (\frac{\partial g}{\partial t} - \frac{\partial f}{\partial u}) \frac{dt}{g}}$$

la ecuación se hace exacta.

* ecuación lineal

$$\frac{du}{dt} = a(t) + b(t)u$$

$$\rightarrow \frac{du}{dt} - b(t)u = a(t)$$

$$L[u] = a(t)$$

* Variación de parámetros

Sea $u(t) = u_h(t) c(t)$ donde $L[u_h] = 0$

$$\rightarrow L[u] = u_h' c + u_h c' - b(u_h c)$$

$$\underbrace{(u_h' - b u_h)}_0 c + u_h c' = a(t)$$

$$\rightarrow c = \int \frac{a(t)}{u_h} dt + c_0$$

$$\rightarrow u = u_h \cdot \left(\int \frac{a}{u_h} dt + c_0 \right)$$

$$u(t) = e^{\int b dt'} \left[u_0 + \int_{t_0}^t e^{-\int b dt'} a(t'') dt'' \right]$$

$$= K(t, t_0) u + \int K(t, t') a(t') dt'$$

teorema de Picard

$$\frac{du}{dt} = f(t, u) \quad / \quad u(t_0) = u_0.$$

$$\text{Si } h = \min\left(a, \frac{b}{M}\right), \quad \left\{ \begin{array}{l} M = \max |f|_{\mathcal{D}} \\ |t - t_0| \leq a \\ |u - u_0| \leq b \end{array} \right\} \mathcal{D}$$

y f cumple con Lipschitz: $|f(t, u_1) - f(t, u_2)| \leq L |u_1 - u_2|$

$\rightarrow \exists! u(t)$ en $\mathcal{D} \mid |t - t_0| \leq h$

Método

Definimos $\{\phi_n\}$

$$\phi_0 = u_0$$

$$\phi_1 = u_0 + \int_{t_0}^t f(t', u_0(t')) dt'$$

$$\vdots$$

$$\phi_n = u_0 + \int_{t_0}^t f(t', \phi_{n-1}(t')) dt'$$

$$u' = \frac{a u}{f(t, u)} \quad / \quad u(0) = 1 \quad (u(t) = e^{at})$$

$$\phi_0 = 1$$

$$\phi_1 = 1 + \int_0^t a dt' = 1 + at$$

$$\phi_2 = 1 + \int_0^t a(1 + at') dt' = 1 + at + \frac{a^2 t^2}{2}$$

$$\downarrow h$$

$$\sim \sum_{n=1}^{\infty} \frac{(at)^n}{n!}$$

SISTEMAS

$$u_1' = a_{11}(t)u_1' + \dots + a_{1n}(t)u_n' + f_1(t)$$

$$u_2' =$$

$$u_n' =$$

$$\Leftrightarrow \begin{pmatrix} u_1' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \quad // \quad \vec{u}(t_0) = \vec{u}_0$$

$$\vec{u}' = A(t)\vec{u} + \vec{f}(t) \Leftrightarrow \underbrace{\left(\frac{d}{dt} - A(t)\right)}_{L[A]} u = f(t)$$

Teorema Si $A(t), f(t)$ son cont. en $(t-t_0) \leq h \rightarrow \exists! \vec{u}(t)$.

① homogéneo: $\vec{u}' = A(t)\vec{u}$

Sea $u(t) = ([u_1] \dots [u_n])$ matriz solución / $u(t_0) = I$

② Leva de Liouville: $\det(u(t)) = |u(t)| e^{\int_{t_0}^t \text{tr}(A(t)) dt}$

$$\rightarrow \frac{d}{dt} u = A(t)u$$

* además, si Y es mat. de soluciones, $Y = u \cdot C$, $C = (c_{ij})$

$$\left(\frac{d}{dt} u^{-1} Y\right) = 0$$

③ Sistemas inhomogéneos

$$\frac{d\vec{u}}{dt} = A(t)u + \vec{f}(t)$$

$$\frac{d\vec{u}}{dt} = A(t)u$$

$$(\vec{u}^{-1}u)' = -\vec{u}^{-1}u' \vec{u}^{-1}u + \vec{u}^{-1}u'$$

$$= -\underbrace{\vec{u}^{-1}A(t)u}_{\vec{u}^{-1}A(t)u} + \underbrace{\vec{u}^{-1}(A(t)u + \vec{f}(t))}_{\vec{u}^{-1}A(t)u + \vec{u}^{-1}\vec{f}(t)} = \vec{u}^{-1}\vec{f}$$

$$\rightarrow (\bar{u}^{-1} \bar{u})' = \bar{u}' \bar{f}$$

$$\rightarrow \bar{u}^{-1}(t) \bar{u}(t) = \bar{u}^{-1}(t_0) \bar{u}(t_0) + \int_{t_0}^t \bar{u}^{-1}(t') \bar{f}(t') dt'$$

$$\rightarrow \bar{u}(t) = \underbrace{\bar{u}(t) \bar{u}^{-1}(t_0)}_{K(t, t_0)} \bar{u}_0 + \int_{t_0}^t \underbrace{\bar{u}(t) \bar{u}^{-1}(t')}_{K(t, t')} \bar{f}(t') dt'$$

$$\frac{d^n u}{dt^n} + a_{n-1} \frac{d^{n-1} u}{dt^{n-1}} + \dots + a_1 \frac{du}{dt} + a_0 u = 0$$

$$\Rightarrow d^n u = -a_{n-1} u^{(n-1)} - \dots - a_0 u$$

$$\begin{pmatrix} u \\ \vdots \\ u^{(n-1)} \end{pmatrix}' = \begin{pmatrix} u' \\ \vdots \\ u^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_{n-1} \end{pmatrix} \begin{pmatrix} u \\ \vdots \\ u^{(n-1)} \end{pmatrix}$$

A

$$\underline{A = d\bar{u}}$$

$$\underline{\frac{d\bar{u}}{dt} = A\bar{u}}$$

/ $t_0 = 0$ (hay invarianza de t
pues $t \rightarrow t - t_0$)

$$\frac{d}{dt} = \frac{d}{d(t-t_0)}$$

Teorema: La matriz fundamental de $\frac{d\bar{u}}{dt} = A\bar{u}$
es e^{At}

$$e^{At} = I + At + \frac{(At)^2}{2!} + \dots$$

Lo mejor es diagonalizar A :

$$A = S \Lambda S^{-1}, \quad (\Lambda)_{ij} = \delta_{ij} \lambda_i$$

Si los autovalores son distintos.

$$I = S \cdot S^{-1} \rightarrow e^{A't} = \sum e^{\lambda_i t} \delta_{ij}$$

Si hay autovalores repetidos,

$$e^{\tilde{A}t} = \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & e^{\lambda_3 t} & \\ & & & e^{\lambda_4 t} \end{pmatrix}$$

$$D_j = \lambda_j \delta_{ij} + J_j$$

$$J_j = \begin{pmatrix} 0 & 1 & 0 & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

$$\underline{u(t)} = e^{A(t-t_0)} u_0 + \int_0^t e^{A(t-\tau)} f(\tau) d\tau$$

$$\text{Si } \frac{d\bar{u}}{dt} = A\bar{u} \rightarrow \left(\frac{d\bar{u}}{dt} - A\bar{u} \right) = 0$$

$$\rightarrow \mathcal{L}[u] = 0$$

homogénea

$$\text{Si } \mathcal{L}[u] = \delta(t)$$

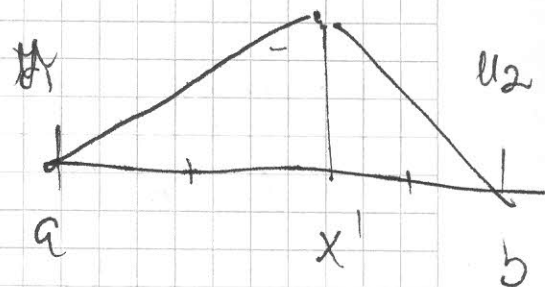
$$\rightarrow G(t, t') = e^{A(t-t')} H(t) = e^{At} H(t)$$

- $$\left\{ \begin{array}{l} 1) \text{ Es not fund en el origen} \\ 2) \text{ Hay salto en la derivada} \\ 3) \lim_{t \rightarrow 0^+} G(t) = I \end{array} \right.$$

~~2~~ A22 STURM-LIOUVILLE

$$-(p(x) \frac{du}{dx})' + q(x)u = f(x)$$

$$\begin{cases} a & B_1: A u(a) + B u'(a) = 0 \\ b & B_2: C u(b) + D u'(b) = 0 \end{cases}$$



→ la sol nia

$$u(x) = \int_a^b G(x, x') f(x') dx'$$

$$\boxed{L[G] = \delta(x) + C.C.} \quad + \frac{dG}{dx} = \frac{1}{p(x')}$$

Supongamos 2 soluciones u_1, u_2 que cumple B_1 y B_2 respect.

→ propongo $G(x, x') = C_1(x') u_1(x) + C_2(x') u_2(x)$

en a $B_1 G(a) = 0$

propongo

$$G(x, x') = \begin{cases} C_1(x') u_1(x) + C_2(x') u_2(x) & a \leq x \leq x' \leq b \\ \bar{C}_1(x') u_1(x) + \bar{C}_2(x') u_2(x) & a \leq x' \leq x \leq b \end{cases}$$

$$B_1 G: A [C_1(x') u_1(a) + C_2(x') u_2(a)] + B [C_1(x') u_1'(a) + C_2(x') u_2'(a)] = 0$$

$$\underbrace{B_1[u_1]}_{=0} + B_1[u_2] = 0 \Leftrightarrow C_2(x') = 0$$

$$B_2 G: C [C_1(x') u_1(b) + C_2(x') u_2(b)] + D [C_1(x') u_1'(b) + C_2(x') u_2'(b)] = 0$$

$$\underbrace{B_2[u_1]}_{=0} + B_2[u_2] = 0 \Leftrightarrow C_1(x') = 0$$

→ $G(x, x') = \begin{cases} C_1(x') u_1(x) & a \leq x \leq x' \leq b \\ C_2(x') u_2(x) & a \leq x' \leq x \leq b \end{cases}$

+ CONT en x' $C_1(x') u_1(x') - C_2(x') u_2(x') = 0$

$$\rightarrow C_2(x') = \frac{C_1(x') u_1(x')}{u_2(x')}$$

+ Disc de $\frac{dS}{dx}$ en x'

$$\left. \frac{dS}{dx} \right|_{x^-}^{x^+} = C_2(x') u_2'(x') - C_1(x') u_1'(x') = -\frac{1}{p(x')}$$

$$= C_1(x') \frac{u_1'(x')}{u_2(x')} - C_1(x') u_1'(x')$$

$$= C_1(x') \left[\frac{u_1(x') u_2'(x') - u_2(x') u_1'(x')}{u_2(x')} \right] = C_1(x') \left[\frac{u_1(x') u_2'(x') - u_2(x') u_1'(x')}{u_2(x')} \right] = -\frac{1}{p}$$

$$\rightarrow C_1 = -\frac{u_2}{p} \frac{1}{u_1 u_2' - u_1' u_2} = -\frac{u_2}{p} \frac{1}{w}$$

$$w = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \neq 0$$

$$\rightarrow C_2 = C_1 \frac{u_1}{u_2} = -\frac{u_2}{p w} \cdot \frac{u_1}{u_2} = -\frac{u_1}{p w}$$

$$\rightarrow G(x, x') = \begin{cases} -\frac{1}{p w(x')} u_1(x) u_2(x') & a \leq x < x' \leq b \\ -\frac{1}{p w(x')} u_1(x') u_2(x) & a \leq x' \leq x \leq b. \end{cases}$$

$$\rightarrow u = \int f(x') G(x, x') dx' \quad (2 \text{ términos}).$$

(*) $p \cdot w = \text{cte}$ pues $L[u_1] u_2 - L[u_2] u_1 = [p w]' = 0$

(*) $G(x, x') = G(x', x)$

(*) L es autoadjunto. Sin soluciones sin en comp completo.

Autoadjunto: $(u, L[v]) = (L[u], v)$

$$L = L^+$$

$$\rightarrow (u, L[v]) - (L[u], v) = 0.$$

$$\int_a^b (u)' v \, dx$$



$$B_1: c_1 u(a) + c_2 u'(a) = 0$$

$$B_2: c_3 u(b) + c_4 u'(b) = 0.$$

No cualquiera: estas

$$\rightarrow 1) G(x, x') = G(x', x)$$

2) Las autofunciones de L son un conjunto completo que son base de las funciones en el dominio.

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Serie de Potencias

$$u'' + A(x)u' + B(x)u = 0$$

Teorema: Si A, B son analíticas en $|x| < R$

$$\rightarrow \begin{cases} A(x) = \sum a_n x^n \\ B(x) = \sum b_n x^n \end{cases}$$

$$\rightarrow u(x) = \sum_{n=0}^{\infty} c_n x^n \quad / \text{vale } x \rightarrow x - x_0.$$

$|x| < R$

Teorema Frobenius-Fuchs: Si A tiene como mucho polo de $\sigma = 1$ y B a lo sumo un polo de $\sigma = 2$ en $x=0$

$$\rightarrow u_1 = x^s \sum_{n=0}^{\infty} a_n x^n$$

$$\parallel \begin{cases} s: s(s-1) + A_{-1}s + B_{-2} = 0 \end{cases}$$

$$\parallel s = \frac{1 - A_{-1} \pm \Gamma}{2}, \quad \Gamma = \sqrt{(1 - A_{-1})^2 - 4B_{-2}}$$

$$\text{si } \begin{cases} r \notin \mathbb{Z} \Rightarrow u_2 = x^{s'} \sum a_n x^n \end{cases}$$

$$\begin{cases} r \in \mathbb{Z} \Rightarrow u_2 = c u_1(x) \ln(x) + x^{s'} \sum b_n x^n \end{cases}$$

s' es la otra raíz.

Autovalores y autofunciones de L (S-L)

$$L[N] = \lambda N$$

Teorema: Si L es autoadjunto $\Rightarrow$ sus autofunciones de autovalor $\neq$ son ortogonales.

$$0 = (N_j, L[N_i]) - (L[N_j], N_i) = (\lambda_i - \lambda_j) \underbrace{\int_a^b N_i N_j dx}_{\delta_{ij}}$$

S.L. AUTOADJUNTO
+
CC

1) $\exists \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$

y $(N_i, N_j) = \delta_{ij}$ ortogonales.

2) Toda $u(x)$ en $[a, b]$ con ∂^2 es

$$u(x) = \sum c_n N_n$$

$$\rightarrow c_n = \frac{(N_n, u)_p}{(N_n, N_n)_p}$$

Base ortonormal $Z_n = \frac{N_n}{\|N_n\|}$

$$\rightarrow c_n = (Z_n, u)_p$$

La convergencia
es ~~uniforme~~
puntual

Parcial

$$\|u\|^2 = \sum c_n^2$$

Si $\{N_n\}$ es completo

$$\|u\|^2 \geq \sum c_n^2$$

Si es incompleto.

Función de Green en autofunciones

$$\text{Si } S = \{z_n\} \quad u = \sum c_n z_n, \quad c_n = (z_n, u)_f$$

$$\rightarrow L[u] = f(x) \quad f = \sum f_n z_n, \quad f_n = (z_n, f)_f$$

$$\rightarrow \sum c_n \lambda_n z_n = \sum f_n z_n$$

$$(z_m, \sum c_n \lambda_n z_n)_f = (z_m, \sum f_n z_n)_f$$

$$c_m \lambda_m = f_m$$

$$\rightarrow c_m = \frac{f_m}{\lambda_m} = \frac{(z_m, f)_f}{\lambda_m} \quad \lambda_m \neq 0.$$

$$\rightarrow u = \sum c_n z_n = \sum \frac{(z_n, f)_f}{\lambda_n} z_n = \sum \int_a^b \frac{f(x') z_n(x')}{\lambda_n} dx' z_n(x)$$

$$= \int_a^b \underbrace{\left(\sum_n \frac{f(x') z_n(x') z_n(x)}{\lambda_n} \right)}_{G(x, x')} f(x') dx'$$

unif. $\lambda_n \neq 0$

¿Qué si $\exists \lambda_j = 0$, $L[u_j] = 0$

$$\rightarrow u = u_0 + \dots + u_n + \int G(x, x') f(x') dx'$$

STURM Y SERIES

$$\begin{cases} a = -\pi \\ b = \pi \end{cases}$$

$$L \equiv -\frac{d^2}{dx^2} \quad p=1$$

$$\rightarrow u_0 = 0 \quad u_n(x) = A \cos(nx) + B \sin(nx)$$

$$\rightarrow f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad \begin{cases} a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \end{cases}$$

El intervalo es simétrico

$$\text{Si } f \begin{cases} \text{par} \rightarrow f = \sum a_n \cos \\ \text{imp} \rightarrow f = \sum b_n \sin \end{cases}$$

En las discontinuidades emerge en media + Gibbs.



en $[-p, p]$

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx, \quad a_n = \frac{1}{p} \int_{-p}^p f(x) \begin{cases} \cos\left(\frac{n\pi}{p} x\right) \\ \sin\left(\frac{n\pi}{p} x\right) \end{cases} dx$$

TRANSF. DE FOURIER

$$\text{Si } f(x) = \frac{1}{2}a_0 + \sum_1^{\infty} \left[a_n \cos\left(\frac{n\pi}{p}x\right) + b_n \sin\left(\frac{n\pi}{p}x\right) \right]$$

$$= \frac{1}{2} \frac{1}{p} \int_{-p}^p f(x) dx + \sum_1^{\infty} \left\{ \frac{1}{p} \int_{-p}^p f(x') \cos\left(\frac{n\pi}{p}x'\right) \cos\left(\frac{n\pi}{p}x\right) dx' + \dots \right\}$$

$$= \sum_{-\infty}^{\infty} a_n e^{i \frac{n\pi}{p}x}, \quad a_n = \frac{1}{p} \int_{-p}^p f(x') e^{i \frac{n\pi}{p}x'} dx'$$

$$\text{Si } p \rightarrow \infty \quad f(x) \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{isx} ds \quad \leftarrow C\{\lambda\} = \mathcal{F}_1$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} dx$$

$$\text{Si } f \begin{cases} a) \int_{\mathbb{R}} |f| < \infty \\ b) f \text{ tiene finitas discontinuidades} \\ \text{finitas} \end{cases} \rightarrow \mathcal{F}'(\mathcal{F}(f(x))) = \begin{cases} f(x) & \text{si es cont} \\ \frac{1}{2}(f(x_0^+) + f(x_0^-)) & \end{cases}$$

$$\text{T. Cos : } \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(sx) dx = F_c(s) \quad s > 0$$

$$\text{T. Sen : } \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(sx) dx = F_s(s) \quad s > 0$$

ECUACIONES EN 2 VARIABLES A ORDEN 2

$$L[u] = a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + 2d \frac{\partial u}{\partial x} + 2e \frac{\partial u}{\partial y} + h u = 0.$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 \\ \lambda_2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda_1, \lambda_2 = / a + 2b\lambda + c\lambda^2 = 0$$

$$\lambda_{1,2} = \frac{-2b \pm \sqrt{4b^2 - 4ac}}{2c} = \frac{-b \pm \sqrt{b^2 - ac}}{c}$$

$$\rightarrow L[u] \rightarrow 2[a + b(\lambda_1 + \lambda_2) + c\lambda_1\lambda_2] \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$\frac{4}{c} [ac - b^2] \frac{\partial^2 u}{\partial \xi \partial \eta}$$

HO y
E E en la
P C

$$\rightarrow u(\xi, \eta) = f(\xi) + g(\eta) = f(x + \lambda_1 y) + g(x + \lambda_2 y)$$

a) $b^2 - ac > 0$ HIPERBÓLICA

$$u = f(x + \lambda_1 y) + g(x + \lambda_2 y)$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial s^2} = 0$$

ONDA

b) $b^2 - ac < 0$ ELÍPTICA

$$\lambda_1 = \lambda_2$$

$$\rightarrow \eta = \xi \rightarrow u = f(\xi) + g(\xi)$$

$$\frac{\partial^2 u}{\partial \xi^2} = 0$$

c) $b^2 - ac = 0$ PARABÓLICA

$$\frac{\partial^2 u}{\partial \eta^2} = 0$$

CALOR

$$u_t - u_{xx} = 0$$

$$u = f(\xi) + \eta g(\xi) = f(x + \lambda y) + y g(x + \lambda y)$$

1

ECUACIÓN DEL CALOR

$$u_t - u_{xx} = 0$$

- Barra ∞ / $u(x, 0) = \varphi(x)$

$$u(s, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x, t) e^{isx} dx$$

$$u_{xx} = \frac{1}{\sqrt{2\pi}} \int u_{xx}(x, t) e^{-isx} dx = \dots = -s^2 u(s, t)$$

$$u_t = \int u_t \dots$$

$$\rightarrow u_t = u_{xx}$$

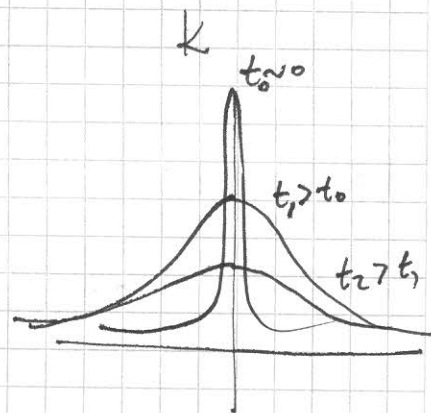
$$u_t = -s^2 u(s, t) \rightarrow u(s, t) = e^{-s^2 t} u(s, 0), \quad u(s, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x') e^{isx'} dx'$$

$$u(s, t) = e^{-s^2 t} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(x') e^{-isx'} dx'$$

$$\rightarrow u(x, t) = \frac{1}{\sqrt{2\pi}} \int e^{isx} e^{-s^2 t} \int \varphi(x') e^{-isx'} dx' ds$$

$$= \int \varphi(x') \int e^{is(x-x')} e^{-s^2 t} ds dx'$$

$$K(x, x') = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$



FUENTES

$$u_t - u_{xx} = f(x, t)$$

$$u(x, 0) = 0 \rightarrow u(s, 0) = 0.$$

$$\rightarrow u_t - s^2 u(s, t) = F(s, t)$$

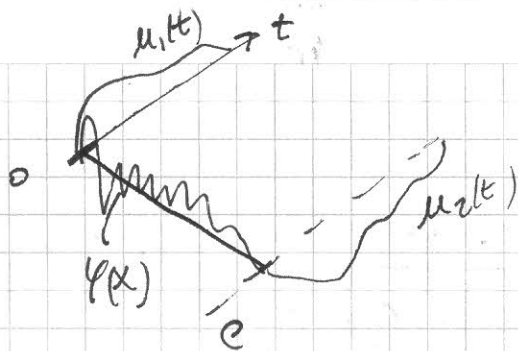
$$\rightarrow u(x, t) = \int \int \frac{1}{\sqrt{4\pi(t-t')}} e^{-\frac{(x-x')^2}{4(t-t')}} H(t-t') f(x', t') dx' dt'$$

$K(x-x', t-t')$

Barrera finita

$$u_t - u_{xx} = 0$$

$$\begin{cases} u(x, 0) = \varphi(x) \\ u(0, t) = u_1(t) \\ u(l, t) = u_2(t) \end{cases}$$



Si $u_1 = u_2 = 0 \rightarrow$ separo variables

$$u = XT$$

$$\rightarrow XT' - X''T = 0 \rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda^2$$

$$T = -\lambda^2 T'$$

$$X = A \cos\left(\frac{n\pi}{l}x\right) + B \sin\left(\frac{n\pi}{l}x\right) \quad \lambda = \frac{n\pi}{l}$$

$$T = C e^{-\left(\frac{l}{n\pi}\right)^2 t}$$

$$u(x, 0) = \sum \tilde{B}_n \sin \frac{n\pi x}{l} = \varphi(x) \rightarrow \tilde{B}_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx$$

PROBLEMA Si $u_t = u_{xx}$

$$u_{\max}(t) \text{ para } t \text{ en } \text{alguna} \begin{cases} t=0 \\ x=0 \\ x=l \end{cases}$$

Barrera semi ∞

$$u_t - u_{xx} = 0$$

$$u(x, t) = \int_{-\infty}^{\infty} \varphi(x') K_0(x \pm x', t) dx'$$



② CUERDA

HIPERBÓLICA

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

D'Alembert $u = f(x+ct) + g(x-ct) = u(x,t)$

$$\begin{cases} u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

$$u(x,0) = f(x) + g(x) = \varphi(x) \rightarrow g = \varphi(x) - f(x) \rightarrow g_t = -f_t(x)$$

$$u_t(x,0) = c f_t(x) - c g_t(x) = \psi(x)$$

$$\rightarrow 2c f_t(x) = \psi(x)$$

↓

$$\rightarrow f - g = \frac{1}{c} \int_a^x \psi(x') dx' \rightarrow f = \frac{1}{c} \int_a^x \psi(x') dx' + g(x)$$

$$\rightarrow 2g(x) + \frac{1}{c} \int_a^x \psi(x') dx' = \varphi(x)$$

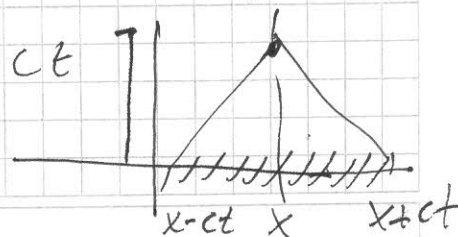
$$\rightarrow g(x) = \frac{1}{2} \left[\varphi(x) - \frac{1}{c} \int_a^x \psi(x') dx' \right]$$

$$\rightarrow f = \frac{1}{c} \int_a^x \psi(x') dx' + \frac{1}{2} \left[\varphi(x) - \frac{1}{c} \int_a^x \psi(x') dx' \right]$$

$$= \frac{1}{2c} \int_a^x \psi(x') dx' + \frac{1}{2} \varphi(x)$$

$$\rightarrow u(x,t) = f(x+ct) + g(x-ct)$$

$$= \frac{1}{2} [\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x') dx' = u_h$$



Cuerda ∞ inhomogénea

$$u_{tt} - c^2 u_{xx} = f(x, t)$$

$$\left. \begin{aligned} u(x, t) &= 0 \\ f(x, t) &= 0 \end{aligned} \right\} t < 0$$

$$u(x, 0) = u_t(x, 0) = 0$$

(Cuerda quieta al principio sin fuerzas o velocidades preexistentes.)

después de proponer

$$\begin{cases} \xi = x + ct \\ \eta = -x + ct \end{cases}$$

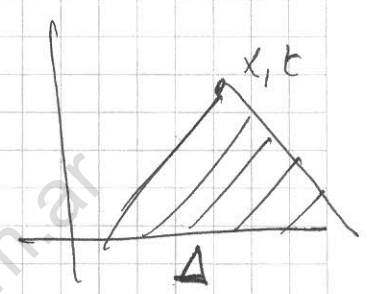
$$u_{tt} - c^2 u_{xx} = f(x, t)$$



$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{1}{4c^2} f\left(\frac{\xi + \eta}{2c}; \frac{\xi - \eta}{2}\right)$$

$$\rightarrow u = \frac{1}{4c^2} \iint f(\xi, \eta) d\xi d\eta$$

$$= \boxed{\frac{1}{2c} \int_{\Delta} \int f(x', t') dx' dt'} = u_p$$



Si es general

$$\rightarrow u(x, t) = u_h + u_p$$

Hölder
(E)
2c

$$\nabla^2 u = 0$$

$$\frac{2D}{\Delta u} = \frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} = 0$$

no es ambigua
compleja si
Dueso

PRO DEL DEL MÁXIMO

Si u es continua en $\mathbb{R} \Rightarrow u$ máximo y mínimo están en $\mathbb{R}$.

CC $\left\{ \begin{array}{l} u|_{\partial\Omega} = f \\ \frac{\partial u}{\partial n} \Big|_{\partial\Omega} = f \end{array} \right.$ Dirichlet
Neumann, $\oint_{\partial\Omega} f \, ds = 0$

Teorema: La Sol Dirichlet para armónicas es única

Tercer: " " " " f " depende continuamente de las condiciones de borde

$$\nabla^2_0 = \partial_r^2 u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial_\phi^2 u = 0.$$

Separate variables, r Euler
 φ cost + Sen

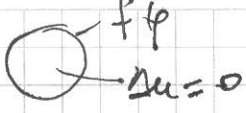
$$\rightarrow u(r, \varphi) = (Cr^1 + Dr^{-1}) (A \cos(\lambda \varphi) + B \sin(\lambda \varphi))$$

en círculo la periodicidad tiene rol gral

$$u(r, \varphi) = A_0 + B_0 \ln(r) + \sum' (C_n r^n + D_n r^{-n}) (A_n \cos(n\varphi) + B_n \sin(n\varphi)).$$

ambros circularas.

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Dirichlet 

$$u(a, \varphi) = f(\varphi) = A_0 + B_0 \ln(a) + \sum_n (C_n a^n + D_n a^{-n}) [A_n \cos(n\varphi) + B_n \sin(n\varphi)]$$

Adentro

$$u(r, \varphi) = \frac{1}{2} A_0 + \sum_n C_n r^n (A_n \cos(n\varphi) + B_n \sin(n\varphi))$$

Afuera =

$$u(r, \varphi) = \frac{1}{2} A_0 + \sum_n D_n r^{-n} (A_n \cos(n\varphi) + B_n \sin(n\varphi))$$

$$u(a, \varphi) = \frac{1}{2} A_0 + \sum_n \underbrace{C_n a^n}_{A_n} \cos(n\varphi) + \underbrace{C_n a^n}_{B_n} \sin(n\varphi) = f(\varphi)$$

$$A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\varphi) d\varphi$$

$$B_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} f(\varphi) \cdot \cos(n\varphi) d\varphi, \quad A_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} f(\varphi) \cdot \sin(n\varphi) d\varphi$$

Teorema de Poisson

$$u(r, \varphi) = \frac{a^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{f(\varphi')}{a^2 + r^2 - 2ar \cos(\varphi - \varphi')} d\varphi'$$

Corolario

$$u(0, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi') d\varphi' = u(0)$$

+ exterior

Weiman

$$\frac{\partial u}{\partial n} = f(\varphi) = \frac{\partial u}{\partial r}$$

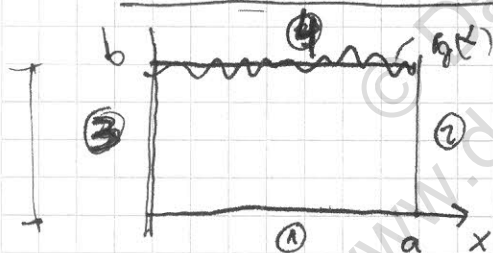
Interior : $u = \frac{A_0 + B_0 \ln(r)}{2} + \sum_1 (C_n r^n + D_n r^{-n}) (A_n \cos(n\varphi) + B_n \sin(n\varphi))$

$$\frac{\partial u}{\partial r} = \sum n C_n r^{n-1} (A_n \cos(n\varphi) + B_n \sin(n\varphi))$$

$$\left. \frac{\partial u}{\partial r} \right|_{r=a} = \sum n C_n a^{n-1} (A_n \cos(n\varphi) + B_n \sin(n\varphi)) = f(\varphi)$$

$$\begin{cases} A_n \\ B_n \end{cases} = \frac{1}{\pi n a^{n-1}} \int_{-\pi}^{\pi} f(\varphi) \begin{cases} \cos(n\varphi) \\ \sin(n\varphi) \end{cases} d\varphi$$

ARMÓNICOS RECTANGULARES



$$u = xy$$

$$\nabla^2 u = x''y + xy'' = 0$$

$$\frac{x''}{x} + \frac{y''}{y} = 0 \rightarrow \frac{x''}{x} = -\frac{y''}{y} = k$$

a) $k=0$ $\begin{cases} x''=0 \rightarrow x(x)=A+Bx \\ y''=0 \rightarrow y(y)=C+Dy \end{cases} \rightarrow u(x,y) = (A+Bx)(C+Dy)$

b) $k=k^2$

$$\begin{cases} X = A \cosh(kx) + B \sinh(kx) \\ Y = C \cos(ky) + D \sin(ky) \end{cases} \quad \begin{cases} u=0 \text{ en } 1, 2, 3 \\ u=g(x) \text{ en } 4 \end{cases}$$

$u(x,0)=0 \rightarrow u(x,b)=0 \Rightarrow C=0 \rightarrow Y=D \sin\left(\frac{n\pi y}{b}\right)$

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$$X = A \cos(kx) + B \sin(kx)$$

$$Y = C \cosh(ky) + D \sinh(ky)$$

$$u(x, y) = [A \cos(kx) + B \sin(kx)] [C \cosh(ky) + D \sinh(ky)]$$

$$u(0, y) = 0$$

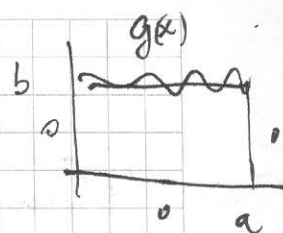
$$u(a, y) = B \sin(ka) [C \cosh(ky) + D \sinh(ky)] = 0 \Leftrightarrow k = \frac{n\pi}{a}$$

$$u(x, 0) = 0 \rightarrow C = 0$$

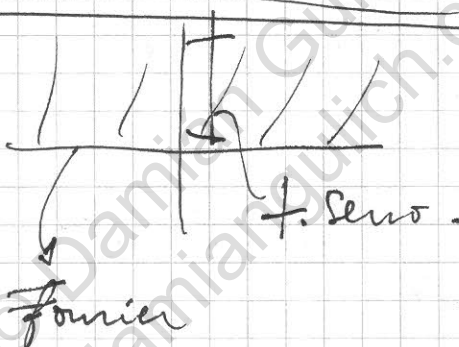
$$u(x, y) = \sum A_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} y\right)$$

$$u(x, b) = \sum A_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left(\frac{n\pi}{a} b\right) = g(x)$$

$$\rightarrow A_n = \frac{1}{\sinh\left(\frac{n\pi}{a} b\right)} \frac{2}{a} \int_0^a g(x') \sin\left(\frac{n\pi}{a} x'\right) dx'$$



Semi-plane



$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x')}{(x-x')^2 + y^2} dx' \quad y > 0.$$

GREEN 1 Gauss a $\iint \phi \nabla \psi \cdot d\vec{A}$

Green 2: Gauss a $\iint (\phi \nabla \psi - \psi \nabla \phi) \cdot d\vec{A}.$

Bessel: $\nabla^2 u = 0$

$$\frac{d^2 u_n}{dr^2} + \frac{1}{r} \frac{du_n}{dr} + \left(k^2 - \frac{n^2}{r^2}\right) u_n = 0$$

$$\rightarrow u = J_n(kr)$$

$$\rightarrow Y_s(z) = \frac{\cos(5\pi) J_s(z) - J_s(z)}{\sin(5\pi)}$$

Legendre

$$\nabla^2 u = 0, \quad u = (z + ix)^n$$

$$P_n(\cos\theta) = P_n^0(\cos\theta)$$

$$P_n(u) = \frac{1}{2^n n!} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n$$

$$P_n^m(\mu) = \frac{(n+m)!}{(n-m)!} \frac{(1-\mu^2)^{-m/2}}{2^n n!} \frac{d^{n-m}}{d\mu^{n-m}} (\mu^2 - 1)^n$$

$$P_n = P_n^0$$

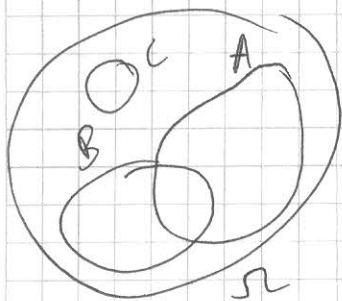
Autofunciones de la esfera.

$$Y_n^m = e^{im\varphi} P_n^m(\mu)$$

$$-n \leq m \leq n$$

PROBABILIDADES

Contar sofisticadamente.

Probabilidad

ACS

$$P: A \mapsto \mathbb{R}$$

$$1) 0 \leq P(A) \leq 1 \quad \forall A$$

$$2) P(\Omega) = 1$$

$$3) \text{ Si } A \cap B = \emptyset \rightarrow P(A \cup B) = P(A) + P(B)$$

$$4) \text{ Si } A_1 \subseteq \dots \subseteq A_{n+1} \subseteq \dots$$

$$\rightarrow P(\cup A_n) = \lim_{n \rightarrow \infty} P(A_n)$$

Si los resultados son equiprobables

$$P(A) = \frac{\text{Card}(A)}{\text{Card}(\Omega)}$$

- Debo hallar $\text{card}(A)$ sin hacer extensión.

$$\text{Dado } A \times B \rightarrow \text{Card}(A \times B) = \text{card}(A) \cdot \text{card}(B)$$

Permutaciones

$$n!$$

Variaciones

$$ab \neq ba$$

$$\binom{n}{k}_a = \frac{n!}{(n-k)!}$$

Arreglos

$$ab = ba$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Paradoja: n personas en cualquier cumpleaños.

$\tilde{p}$ = prob. de cumplir igual distinto

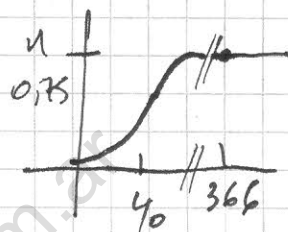
$$\tilde{p} = \frac{\text{card}(A)}{\text{card}(\Omega)}$$

$$\text{card}(\Omega) = 365^n$$

(repetido 365 veces entre n personas)

$$\text{card}(A) = \frac{365!}{n!} \rightarrow \tilde{p} = \frac{365!}{n! \cdot 365^n}$$

$$\rightarrow p = 1 - \tilde{p}$$

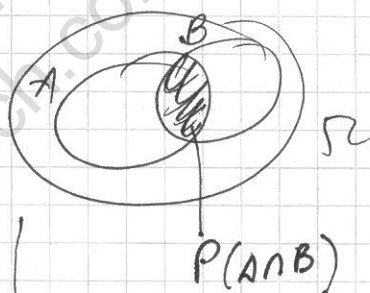


Condicionalidad

$$P(A|B) = P(A \cap B) / P(B)$$

$$\rightarrow \boxed{P(A \cap B) = P(A|B) \cdot P(B)}$$

$$P(A|B) \neq P(B|A)$$



IND

A, B son indep. si $P(A \cap B) = P(A) \cdot P(B)$

MODELOS CONDICIONALES

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_n)$$

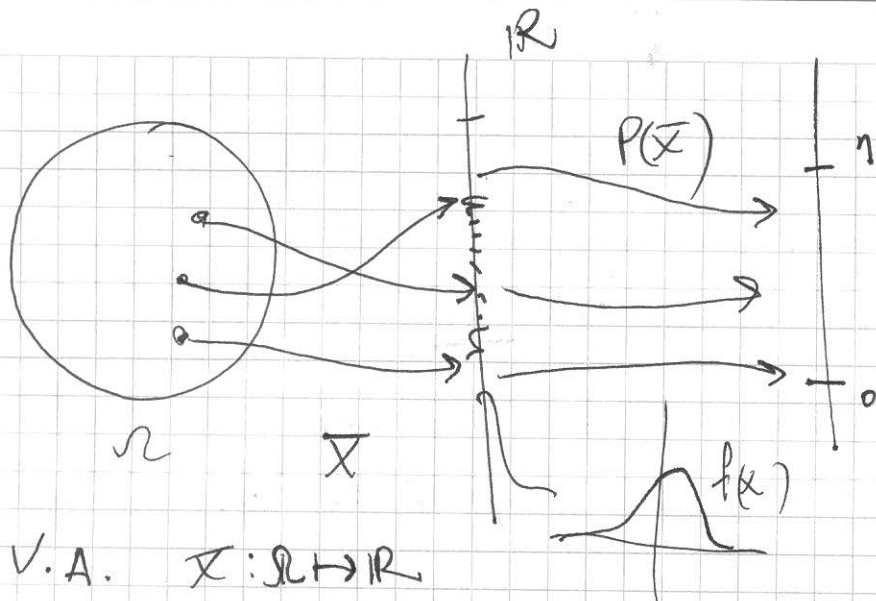
$$\rightarrow P(A) = P(A \cap B_1) + \dots + P(A \cap B_n)$$

$$= \sum (A|B_i) \cdot P(B_i) \quad (1)$$

$$\rightarrow P(B_k|A) = \frac{P(A|B_k) \cdot P(B_k)}{\sum (A|B_i) \cdot P(B_i)}$$

Bayes

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V.A. $X: \Omega \rightarrow \mathbb{R}$ $P(X): \mathbb{R} \rightarrow [0, 1]$

$$p(x) = P(X=x)$$

ACUMULADA $F(x) = P(X \leq x) = \sum_{y \leq x} p(y)$

$$\rightarrow P(a < X \leq b) = F(b) - F(a)$$

$$\mu_x = E(X) = \sum x_i p(x_i) \quad / \quad E(f(x)) = \sum f(x_i) p(x_i)$$

$$\sigma_x^2 = \sum (x_i - \mu)^2 p(x_i)$$

$$DS = \sqrt{\sigma_x^2}$$

dens. de p.

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

$$\mu_x = \int x p(x) = E(X)$$

$$\sigma_x^2 = \int (x - \mu_x)^2 p(x) = E(x - \mu)^2$$

$$\mu_n = E(x^n)$$

$$\sigma_n = E(x - \mu)^n$$

$$a) f(x) \geq 0$$

$$b) \int_{\mathbb{R}} f(x) dx = 1$$

Teorema de Tchebycheff : si $k > 0$
 $\rightarrow P(|\bar{x} - \mu| \leq k\sigma) \leq \frac{1}{k^2}$

Función generatriz

$$K(t) = \langle e^{ixt} \rangle = \int_{\mathbb{R}} e^{ixt} f(x) dx$$

$\rightarrow$ a menos de un $(i)^n \frac{d^n K}{dt^n} \Big|_{t=0} = \mu_n + i^n \mu_n$

centrada

$$K_\mu(t) = \langle e^{i(x-\mu)t} \rangle = e^{-i\mu t} \underbrace{\int_{\mathbb{R}} e^{ixt} f(x) dx}_{K(t)}$$

$$\rightarrow \frac{d^n K_\mu}{dt^n} \Big|_{t=0} = i^n \sigma_n$$

Normal: $f(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}} \sigma e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$E(x_{\text{normal}}) = \mu$$

$$\sigma_{\text{normal}}^2 = \sigma^2$$

$$\rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

La acumulada

$$\Phi(z) = \int_{-\infty}^z e^{-\frac{z^2}{2}} dz = \text{erf}(z)$$

$$K_{\text{norm}} = \langle e^{izt} \rangle$$

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En más dimensiones

$$(x_1, \dots, x_n) \in A \subset \mathbb{R}^n$$

$$f(x_1, \dots, x_n) \geq 0 \quad \forall (x_1, \dots, x_n)$$

$$\underbrace{\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}_n f(x_1, \dots, x_n) dx_1 \dots dx_n = 1$$

$$\rightarrow P[(x_1, \dots, x_n) \in A] = \underbrace{\int_{\text{en } A}} \dots \int dx_n f(x_1, \dots, x_n) = \int_A f(p) dp.$$

$$\text{INDEP: } f(x_1, \dots, x_n) = f(x_1) \cdot f(x_2) \dots f(x_n)$$

$$\begin{aligned} \rightarrow F(x_1, \dots, x_n) &= \int_{-\infty}^{x_1} dx_1' \dots \int_{-\infty}^{x_n} dx_n' f(x_1', \dots, x_n') \\ &= \int_{-\infty}^A da f(a) \end{aligned}$$

2D

Covarianza

$$\boxed{\text{Cov}(X, Y) = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y}}$$

Correlación

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

ESTADÍSTICA

1) - mudo mudo 1 solo VA

- tengo 1 ~~resultados~~ muestra de n resultados.

- todos los V. tienen la misma dispersión

Si 1) Son las n indep

2) tienen la misma dispersión

$$\rightarrow T = \sum X_i$$

$$\rightarrow \langle X \rangle = \mu_x = \frac{\sum x_i}{n}$$

Las pueden combinar linealmente

$$x_1 \dots x_n$$

$$\downarrow$$

$$x_1 \quad \bar{x}_n$$

$$f(x_1 \dots x_n), \quad \bar{Y} = \sum a_i x_i$$

$$\rightarrow \chi_y(t) = \int \dots \int dx_1 \dots dx_n e^{\sum a_j x_j t} f(x_1 \dots x_n)$$

$$\rightarrow E(\bar{Y}) = -i \left. \frac{\partial \chi_y}{\partial t} \right|_{t=0} = \sum a_j E(x_j)$$

$$V(\bar{Y}) = \sum_i \sum_j a_i a_j \frac{E[(x_i - \mu_i)(x_j - \mu_j)]}{\text{cov}(\bar{x}_i, \bar{y}_j)}$$

Si $f(x_1 \dots x_n) = f(x_1) \dots f(x_n)$

a) $\chi_n = \pi \left[\int dx_j e^{i a_j x_j t} f_j(x_j) \right] = \pi \chi_j(a_j t)$

b) $E(\bar{Y}) = \sum a_j E(x_j)$

c) $\text{Var}(\bar{Y}) = \sum a_i^2 V(x_i)$

31/10/17
Si $X_1, \dots, X_n$ son indep. normales pero no neces todas aleat
 $\Rightarrow$ cualquier C.L. de ellas es normal:

$$(X_i) = N(\mu_i, \sigma_i^2), \quad Y = \sum a_i X_i$$

$$\rightarrow (Y) = N\left(\sum a_i \mu_i, \sqrt{\sum a_i^2 \sigma_i^2}\right)$$

Teorema central lín

Si $X_1, \dots, X_n$ son V.A. con cualesquiera distrib., y propios μ_i, σ_i^2 tales que

1) $\frac{\mu_i}{\sigma_i} > \sigma_i^2 \quad i=1, \dots, n$

2) $\sigma_{3i} \neq 0$ (no σ_i es el centrado de orden 3).

$$\Rightarrow \lim_{n \rightarrow \infty} \bar{X} \text{ es } N\left(\mu_x = \sum \frac{\mu_i}{n}, \sigma_x^2 = \frac{\sum \sigma_i^2}{n^2}\right)$$

Ley de Grandes Números

Si $X_1, \dots, X_n$ tienen la misma dist.,

$\Rightarrow$ la media muestral tiende a la media poblacional

$$\begin{cases} \mu_1 = \mu_2 = \dots = \mu \\ \sigma_1^2 = \sigma_2^2 = \dots = \sigma^2 \end{cases}$$

Forma débil: Si X_i son indep. con μ, σ^2 fijos

$$\rightarrow P(|\bar{X}_n - \mu| > \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \rightarrow 0 \quad n \rightarrow \infty$$

Forma fuerte: Si $\exists \mu = E(X_i)$

$$\rightarrow \lim_{n \rightarrow \infty} P(\bar{X}_n \rightarrow \mu) = 1$$

ESTIMADORES

- Cmoza f .

- Propósito que $x_1, \dots, x_n$ dependen de θ desconocido.

→ Def $\hat{\theta}(x_1, \dots, x_n)$ elegida para aproximar a θ con algún criterio

CRITERIOS

1) $E(\hat{\theta}) = \theta$ insesgado

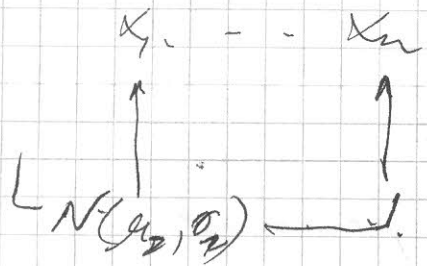
2) $V(\hat{\theta}) \leq V(\tilde{\theta}^*)$, $\tilde{\theta}^* \neq \hat{\theta}$

Máxima verosimilitud

Halla máx. estimadores eficientes

$\propto f(x_1, \dots, x_n, \theta)$

→ $\frac{\partial}{\partial \theta} \ln[f(x_1, \dots, x_n, \theta)] = 0$



- No hay memoria

$$\hat{\theta}(x_1, \dots, x_n, \theta)$$

$$1) E(\hat{\theta}) = \theta$$

$$\mathcal{L}(x_1, \dots, x_n, \theta)$$

$\bar{x}$ es insesgada

$$\mu = \bar{x} \mid \text{muestra} + \sigma$$

$$\hat{\sigma}^2 = \frac{\sum (x - x_i)^2}{n-1}$$

La media muestral - insesgada.